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THE INFLUENCE OF AN EXTERNAL MAGNETIC FIELD ON THE DYNAMIC STRESS OF AN ELASTIC CONDUCTING ONE-SIDED LAYER WITH A LONGITUDINAL SHEAR CRACK

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We study the interaction of a magnetoelastic shear wave with a curvilinear tunnel crack in an ideally conducting diamagnetic (resp. paramagnetic) one-sided (resp. two-sided) layer in the presence of an external static magnetic field. The bases of the one-sided layer are free of mechanical load, and the rim of the face is clamped or free. The corresponding linearized boundary-value problem of magnetoelasticity is reduced to a singular integrodifferential equation with subsequent implementation on a computer. We give numerical results that characterize the influence of the size of the preliminary magnetic field, the frequencies of the load, the curvature, and the orientation of the crack on the stress intensity factor.

3 Figures. Bibliography: 6 titles.

Strong static magnetic fields may exert a determining influence on the diffraction of elastic stress waves in bodies with inhomogeneities. In this situation the mechanical and electromagnetic fields caused by the motion of an elastic diamagnetic (resp. paramagnetic) medium are coupled. This is manifested in the presence of Lorentz forces in the equations of motion and the additional Maxwell stress tensor [1].

Certain diffraction problems of a magnetoelastic wave on a rectilinear crack in an unbounded medium were studied in [2, 3]. In the present paper we study the interaction of a shear magnetoelastic wave with a tunnel curvilinear crack in a diamagnetic (resp. paramagnetic) one-sided layer.

We consider an ideally conducting elastic one-sided layer occupying the region $0 \le x \le a$, $0 \le y < \infty$, $-\infty < z' < \infty$ in xyz'-coordinates. This one-sided layer is located in a static magnetic field of intensity $\vec{H}^0 = (0, H_0, 0)$ and weakened by a cylindrical slit-crack along the z'-axis.

Suppose a magnetoelastic shear displacement wave W_0 radiates from infinity [2], and the surface of the crack is either free of forces or subject to the action of a time-harmonic load that is constant along a generator. In this case a stationary wave process arises in the body corresponding to a state of antiplane deformation: $\vec{U} = (0, 0, We^{-i\omega t})$ (here $\vec{U}$ is the elastic displacement vector, ω is the periodic frequency, and t is time). All quantities that characterize the stress-strain and electromagnetic state of the medium contain the time factor $e^{-i\omega t}$, which we shall omit in what follows.

The tension of the quasistatic electromagnetic field induced in the body can be obtained from the linearized Maxwell equations [1] in the form

$$\vec{h} = (-H_0 W, 0, 0), \quad \vec{e} = (i\mu_e \omega H_0 W, 0, 0). \tag{1}$$

We write the resolvent equation for the displacement $W(x, y)$, which follows from the linearized equations of motion, as follows:

$$\nabla^2 W + \chi^2 \partial^2 W/\partial y^2 + \gamma_2^2 W = 0 \quad (\chi^2 = \mu_e H_0^2/\mu, \quad \gamma_2^2 = \rho\omega^2\mu). \tag{2}$$

Here μ_e is the magnetic permeability, ρ is the density of the medium, μ is the shear modulus, γ_2 is the wave number, and ∇^2 is the Laplacian.

The total stresses σ_{xz}, σ_{yz} are obtained by adding the mechanical stresses τ_{xz} and τ_{yz} and the Maxwell stresses t_{xz}, t_{yz} and are expressed in terms of the displacement $W(x, y)$ according to the formulas

$$\sigma_{xz} = \tau_{xz} + t_{xz}, \ \ \sigma_{yz} = \tau_{yz} + t_{yz}, \ \ \tau_{xz} = \mu\partial W/\partial x, \ \ \tau_{yz} = \mu\partial W/\partial y, \ \ t_{xz} = 0, \ \ t_{yz} = \mu\chi^2\partial W/\partial y. \tag{3}$$

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Assume that the bases of the one-sided layer are free of forces

$$\partial W/\partial x = 0 \quad (x = 0, x = a). \tag{4}$$

We give the condition on the face boundary of the one-sided layer in the form

$$A(A-1)W + A(A+1)\partial W/\partial y = 0 \quad (y = 0). \tag{5}$$

Here if $A = 1$ or -1, we have respectively a free or a reinforced face rim for the one-sided layer; if $A = 0$, we are considering a layer $0 \le x \le a$, $-\infty < y < \infty$, $-\infty < z' < \infty$.

We assume that a magnetoelastic shear wave

$$W_0 = \tau \exp\{-i\gamma_2 y/\sqrt{1+\chi^2}\}, \tag{6}$$

is propagating along the negative direction of the y-axis, and on the edges of the crack S a time-harmonic mechanical load $X_n^\pm = Y_n^\pm = 0$, $Z_n^\pm = \pm Z$ is possible.

Let L be the line of intersection of the surface S and the xy-plane, and let $\vec{n} = (\cos\psi, \sin\psi)$ be the unit normal to L. Assume that Z and the curvature of the arc L are functions of class H [4]. We write the boundary condition on the edge of the crack in the form

$$(\partial W/\partial n)\big|_L = \frac{1}{\mu} z. \tag{7}$$

The equation of the vibrations (2) in the passage to new coordinates $x_1 = x$, $y_1 = y/\sqrt{1+\chi^2}$ can be transformed into the Helmholtz equation

$$\partial^2 W/\partial x_1^2 + \partial^2 W/\partial y_1^2 + \gamma_2^2 W = 0. \tag{8}$$

The differentials ds_1 and ds of arc length on L in the coordinate systems x_1y_1 and xy are connected by the relation $(1+\chi^2)\, ds_1^2 = (1+\chi^2\sin^2\psi)\, ds^2$.

We represent the solution of the problem (2), (4)–(7) in the form of a superposition of the incident, reflected, and scattered waves:

$$\begin{gathered}
W = W_0 + AW_1 + W_*, \quad W_1 = \tau\exp(i\gamma_2 y_1), \\
W_*(x,y) = -i\int_L \Big\{p(s)\Big(\frac{\partial G}{\partial \zeta_1}\, d\zeta_1 - \frac{\partial G}{\partial \bar\zeta_1}\, d\bar\zeta_1\Big) - \frac{i\chi^2 \sin 2\psi}{2\sqrt{1+\chi^2}} p'(s) G\, ds\Big\}, \\
G(\xi, x, \eta_1, y_1) = g(\xi, x, \eta_1 - y_1) + Ag(\xi, x_1, \eta_1 - y_1), \\
g(\xi, x, \eta_1 - y_1) = \frac{1}{2i\gamma_2 a} e^{i\gamma_2|\eta_1 - y_1|} - \frac{1}{a}\sum_{k=1}^{\infty} \frac{1}{\lambda_k} e^{-\lambda_k|\eta_1 - y_1|}\cos\alpha_k\xi\cos\alpha_k x, \\
\lambda_k = \sqrt{\alpha_k^2 - \gamma_2^2} \quad (\gamma_2 < \alpha_k), \quad \lambda_k = -i\sqrt{\gamma_2^2 - \alpha_k^2} \quad (\gamma_2 > \alpha_k), \quad \alpha_k = \frac{\pi k}{a}, \\
\zeta_1 = \xi + i\eta_1, \quad \eta_1 = \eta/\sqrt{1+\chi^2}, \quad \zeta = \xi + i\eta \in L, \quad z = x + iy.
\end{gathered} \tag{9}$$

Here $p(s)$ is the unknown density, W_1 is the wave reflected from the face boundary of the one-sided layer, G is the Green's function of the boundary-value problem (8), (4), (5) for the semi-infinite strip $0 \le x_1 \le a$, $0 \le y_1 < \infty$, and g is the Green's function for the boundary-value problem (8), (4) for the strip $0 \le x_1 \le a$, $-\infty < y_1 < \infty$.

The integral representation (9) satisfies the differential equation (2), the boundary conditions (4), (5), and the radiation conditions of [5], and also guarantees the continuity of the mechanical stresses and the existence of a jump in displacement on the contour L.

To exhibit the logarithmic singularity of the Green's function and the strengthening of the convergence of the series in (9) we sum its principal part. The result is

$$g(\xi, x, \eta_1 - y_1) = \frac{1}{2\pi}\ln\left|4\sin\frac{\pi}{2a}(\zeta_1 - z_1)\sin\frac{\pi}{2a}(\bar{\zeta}_1 + z_1)\right| - \frac{1}{2a}|\eta_1 - y_1)| + \frac{1}{2i\gamma_2 a}e^{i\gamma_2|\eta_1 - y_1|}$$
$$- \frac{1}{a}\sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}e^{-\lambda_k|\eta_1 - y_1|} - \frac{1}{\alpha_k}e^{-\alpha_k|\eta_1 - y_1|}\right)\cos\alpha_k\xi\cos\alpha_k x, \quad z_1 = x + iy_1. \quad (10)$$

We compute the normal derivative of the function W of (9) by regularizing the divergent integrals through integration by parts and repeatedly summing the principal parts of the series for the second derivatives of the Green's function. Substituting the limiting value of the normal derivative as $z \to \zeta_0 = \xi_0 + i\eta_0 \in L$ into the boundary condition (7), we arrive at a singular integro-differential equation on L with respect to the function $p(s)$:

$$\int_L \{p'(s)h(s, s_0) + p(s)H(s, s_0)\}\,ds = F(s_0), \quad (11)$$

$$h(s, s_0 = \operatorname{Im}\frac{c(\psi_0)}{\zeta_1 - \zeta_0}, + \frac{1}{8a}\chi^2\chi_1\sin 2\psi\{\operatorname{Re}[c(\psi_0)(\cot\zeta_2 - \cot\zeta_3 + 2i\operatorname{sgn}\eta_2)]$$
$$- 2\chi_1\sin\psi_0(\operatorname{sgn}\eta_2 \cdot e^{-i\gamma_2|\eta_2|} - Ae^{i\gamma_2\eta_3}) - 4\sum_{k=1}^{\infty} c_{1k}(\alpha_k\varphi_{1k}s_{2k}\cos\psi_0 - \chi_1\varphi_{2k}c_{2k}\sin\psi_0)\},$$

$$H(s, s_0) = -\frac{\pi}{8a^2}\operatorname{Im}\{c'(\psi)[c(\psi_0)(\zeta_2^{-2} - \csc^2\zeta_2) + \overline{c(\psi_0)}\csc^2\zeta_3]\}$$
$$+ \frac{1}{4\pi}\gamma_2^2\chi_1\left[\cos(\psi - \psi_0)\ln|\sin\zeta_2| - \cos(\psi + \psi_0)\ln|\sin\zeta_3| + \left(2\ln 2 - \frac{\pi}{a}|\eta_2|\right)\sin\psi\sin\psi_0^*\right]$$
$$- \frac{i\gamma_2\chi_1}{2a}\sin\psi\sin\psi_0(e^{-i\gamma_2|\eta_2|} + Ae^{i\gamma_2\eta_3}) + \frac{\chi_1}{a}\sum_{k=1}^{\infty}[\varphi_{3k}c_{1k}c_{2k}\sin\psi\sin\psi_0 - \alpha_k^2\varphi_{1k}s_{1k}s_{2k}\cos\psi\cos\psi_0 +$$
$$\alpha_k\varphi_{2k}(\chi_1 s_{1k}c_{2k}\cos\psi\sin\psi_0 - \chi_0 c_{1k}s_{2k}\sin\psi\cos\psi_0) + \frac{\gamma_2^2}{\alpha_k}(s_{1k}s_{2k}\cos\psi\cos\psi_0 + c_{1k}c_{2k}\sin\psi\sin\psi_0)e^{-\alpha_k|\eta_2|}],$$

$$F(s_0) = \frac{1}{\mu}Z + \frac{\tau\chi_1}{2a}\sin\psi_0(e^{-i\gamma_2 y_1} - Ae^{i\gamma_2 y_1}), \quad p(s) = \int_0^s p'(s)\,ds, \quad c(\psi_0) = \cos\psi_0 + i\chi_1\sin\psi_0,$$
$$c'(\psi_0) = -\sin\psi_0 + i\chi_1\cos\psi_0,$$
$$\varphi_{1k} = \frac{1}{\lambda_k}(e^{-\lambda_k|\eta_2|} + Ae^{-\lambda_k\eta_3}) - \frac{1}{\alpha_k}e^{-\alpha_k|\eta_2|}, \quad \varphi_{2k} = \operatorname{sgn}\eta_2 \cdot (e^{-\lambda_k|\eta_2|} - e^{-\alpha_k|\eta_2|}) - Ae^{-\lambda_k\eta_3},$$
$$\varphi_{3k} = \lambda_k(e^{-\lambda_k|\eta_2|} + Ae^{-\lambda_k\eta_3}) - \alpha_k e^{-\alpha_k|\eta_2|},$$
$$\zeta_2 = \frac{\pi}{2a}(\zeta_1 - \zeta_{01}), \quad \zeta_3 = \frac{\pi}{2a}(\bar{\zeta}_1 + \zeta_{01}), \quad \eta_2 = \eta_1 - \eta_{01}, \quad \eta_3 = \eta_1 + \eta_{01},$$
$$\zeta_{01} = \xi_0 + i\eta_{01}, \quad \eta_{01} = \chi_1\eta_0, \quad \chi_0 = \sqrt{1+\chi^2}, \quad \chi_1 = 1/\sqrt{1+\chi^2},$$
$$c_{1k} = \cos\alpha_k\xi, \quad c_{2k} = \cos\alpha_k\xi_0, \quad s_{1k} = \sin\alpha_k\xi, \quad s_{2k} = \sin\alpha_k\xi_0.$$

Here the kernel $h(s, s_0)$ is singular, and $H(s, s_0)$ has a logarithmic singularity.

The condition that there are no discontinuities in displacement at the tips of the slit leads to the equality

$$\int_L p'(s)\,ds = 0. \quad (12)$$

Equations (11) and (12) determine uniquely the solution with an unbounded derivative at the tips of the slit [4].

It is convenient in what follows to introduce a parametrization of the contour of the slit: $\zeta = \zeta(\beta)$ $(-1 \le \beta \le 1)$. Accordingly $p'(s) = \Omega(\beta)/(s'(\beta)\sqrt{1-\beta^2})$, $\Omega(\beta) \in H[-1, 1]$.

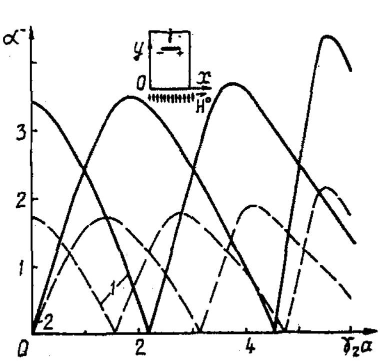

Fig. 1

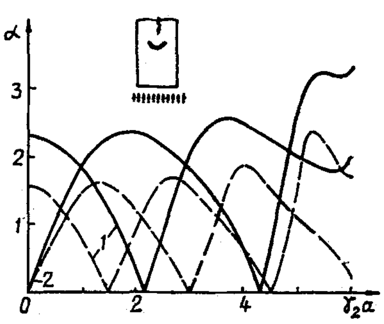

Fig. 2

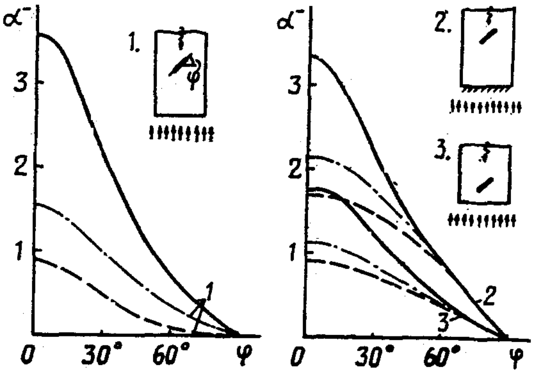

Fig. 3

To determine the total stresses σ_{xz} and σ_{yz} in a neighborhood of the tip of the defect we use the integral representation (9). Asymptotic analysis of the integrals occurring in the formulas for the stresses yields:

$$\sigma_{xz} - i\sigma_{yz} = -\frac{\mu}{2}\sqrt{1+\chi^2}e^{i\psi_c/2}\Omega(\pm 1)/\sqrt{\mp 2is'(\pm 1)(z-c)} + O(1), \quad z \to c, \quad c = \zeta(\pm 1), \quad \psi_c = \psi\big|_{\tau=c}. \quad (13)$$

Taking account of (13), we determine the dynamic stress intensity factor

$$K_{111} = \lim_{r\to 0}\sqrt{2\pi r}(\sigma_{xz}\cos\psi_r + \sigma_{yz}\sin\psi_c) = \pm\frac{\mu}{2}\sqrt{\pi(1+\chi^2)/s'(\pm 1)}\Omega(\pm 1), \quad (14)$$

where r is the distance from the point in question to the extension of the crack up to the face c.

A numerical implementation of Eqs. (11), (12) was carried out by the method of mechanical quadratures using the Gauss-Chebyshev quadrature formulas for regular and singular integrals [6].

Computations of the dimensionless quantities $\alpha^\pm$ were carried out for the case of diffraction of an incident shear wave ($\tau \neq 0$) on a load-free crack ($Z = 0$). The stress intensity factor can be expressed in terms of $\alpha^\pm$ as follows: $K_{111} = \sqrt{a}\alpha^\pm|T_y|\arg\Omega(\pm 1)$, $|T_y| = \mu\gamma_2\tau/\sqrt{1+\chi^2}$ is the modulus of mechanical stress in the incident wave, the upper sign corresponding to the face $\zeta(1)$ of the crack and the lower to $\zeta(-1)$.

Figures 1 and 2 depict the dependence of the quantity α^- on the normalized wave number $\gamma_2 a$. The parametric representation of the contour of the crack is the following: $\xi/a = 0.5 + 0.2\beta$, $\eta/a = 1 + p\beta^2$, $-1 \leq \beta \leq 1$. The value of the curvature parameter $p = 0$ corresponds to a rectilinear crack (Fig. 1), $p = 0.1$ to a parabolic crack (Fig. 2). Curves *1* are constructed for a clamped face rim of the one-sided layer ($A = -1$), curves *2* to a free face ($A = 1$). The solid curves correspond to the value $x = 1$ and the dashed curves to $x = 0$. It can be seen that the applied magnetic field shifts the extreme points toward the larger values of $\gamma_2 a$. Here for a given distance from the rectilinear crack to the face rim the quantity K_{111} vanishes for $\gamma_2 a = (2k+1)\pi\sqrt{1+\chi^2}/2$, $A = -1$ and $k\pi\sqrt{1+\chi^2}$, $A = 1$ ($k = 0, 1, \ldots$). As $\gamma_2 a$ is increased for a layer ($A = 0$) the quantity α^- increases very little.

Figure 3 shows the variation of the quantity α^- from the angle of inclination φ of the rectilinear crack of length $0.4a$ for a one-sided layer with a clamped face ($A = 1$, $\gamma_2 a = 4$, curves *1*) and a free face ($A = 1$, $\gamma_2 a = \pi/2$, curves *2*), and for a layer ($A = 0$, $\gamma_2 a = 4$, curves *3*). The equations of the contour L have the form $\xi/a = 0.5 + 0.2\beta\cos\varphi$; $\eta/a = 1 + 0.2\beta\sin\varphi$, $-1 \leq \beta \leq 1$. The solid curves correspond to the value $\chi = 1$, the dash-dotted lines to $\chi = 0.5$, and the dashed lines to $\chi = 0$. The nature of the influence of the magnetic field on the stress intensity factor K_{111} depends both on the angle of inclination of the crack and the form of the boundary condition at the face $y = 0$. Thus, an applied static magnetic field corresponding to the value $\chi = 1$ significantly increases α^- when $\varphi \in [0, 45°]$ for all the values of the parameter A considered. At the same time for $\varphi \in [45°, 90°]$ the quantity α^- increases sharply in the case of a clamped face and very little in the case of a layer and a one-sided layer with a load-free face.

The dependences represented illustrate the possibility of controlling the dynamic stress of the body using an external magnetic field.

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